

# On the number of Courant-sharp Dirichlet eigenvalues

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## Abstract

We consider arbitrary open sets  $\Omega$  in Euclidean space with finite Lebesgue measure, and obtain upper bounds for (i) the largest Courant-sharp Dirichlet eigenvalue of  $\Omega$ , (ii) the number of Courant-sharp Dirichlet eigenvalues of  $\Omega$ . This extends recent results of P. Bérard and B. Helffer.

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## 1 Introduction

Let  $\Omega$  be an open set in Euclidean space  $\mathbb{R}^m$  with finite Lebesgue measure  $|\Omega|$  and boundary  $\partial\Omega$ . We denote the spectrum of the Dirichlet Laplacian acting in  $L^2(\Omega)$  by  $\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots$  taking the multiplicities of these eigenvalues into account. We define the counting function for  $\Omega$  by

$$N_\Omega(\lambda) = \#\{n \in \mathbb{N} : \lambda_n(\Omega) < \lambda\}.$$

Weyl's law asserts that

$$N_\Omega(\lambda) = \frac{\omega_m}{(2\pi)^m} |\Omega| \lambda^{m/2} + o(\lambda^{m/2}), \lambda \rightarrow \infty, \quad (1)$$

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where  $\omega_m$  is the measure of a ball  $\mathcal{B}_m$  with radius 1 in  $\mathbb{R}^m$ . We refer to Theorem 2 in [16] for a proof of (1) in this generality. For a proof of Weyl's law with a non-trivial remainder estimate for  $\Omega$  open, bounded and connected we refer to Theorem 1.8 in [12].

Let  $\{\varphi_{1,\Omega}, \varphi_{2,\Omega}, \dots\}$  be an orthonormal basis in the Sobolev space  $H_0^1(\Omega)$  of eigenfunctions corresponding to the Dirichlet eigenvalues. These eigenfunctions satisfy the Dirichlet boundary conditions in the usual trace sense. Let  $\nu(\varphi_{n,\Omega})$  denote the number of nodal domains of  $\varphi_{n,\Omega}$ . Then Pleijel's theorem ([13]) states that

$$\limsup_{n \rightarrow \infty} \frac{\nu(\varphi_{n,\Omega})}{n} \leq \gamma_m,$$

where

$$\gamma_m = \frac{(2\pi)^m}{\omega_m^2} (\lambda_1(\mathcal{B}_m))^{-m/2} < 1. \quad (2)$$

It is known that Pleijel's bound is not sharp. See [7], [18] and [14].

We say that  $\lambda_n(\Omega)$  is Courant-sharp if  $\nu(\varphi_{n,\Omega}) = n$ . Courant's nodal domain theorem asserts that  $\nu(\varphi_{n,\Omega}) \leq n$ . Courant's original proof in [8] was for the planar case. This has been subsequently stated and proved in a Riemannian manifold setting in [3]. See also [13]. Pleijel's theorem implies that for a given  $\Omega$  the number of Courant-sharp Dirichlet eigenvalues is finite. Using results of [5] and [17], Bérard and Helffer, [1], obtained an upper bound for the largest Courant-sharp Dirichlet eigenvalue if  $\Omega$  is bounded and has smooth boundary  $\partial\Omega$ .

This paper concerns arbitrary open sets in  $\mathbb{R}^m$  with finite Lebesgue measure. The proofs of Courant's theorem in [8], [13] and [3]) all use the fact that a restriction of an eigenfunction to a nodal domain  $U$  is the first Dirichlet eigenfunction on  $U$ . This is immediate if  $(\partial\Omega) \cap (\partial U)$  is sufficiently regular. The above fact holds without that regularity requirement. See for example Theorem 1.1 in [9].

Our main result, Theorem 1 below is for open sets  $\Omega$  in  $\mathbb{R}^m$  with finite Lebesgue measure. We obtain (i) an upper bound for the largest Dirichlet eigenvalue of  $\Omega$  which is Courant-sharp, and (ii) an upper bound for the number of Courant-sharp eigenvalues of  $\Omega$ . For  $A \subset \mathbb{R}^m, A \neq \emptyset$  let

$$d(x, A) = \inf\{|x - y| : y \in A\}.$$

For  $\epsilon \geq 0$  and  $|\Omega| < \infty$  we define

$$\mu_\Omega(\epsilon) = |\{x \in \Omega : d(x, \partial\Omega) < \epsilon\}|,$$

and

$$\epsilon(\Omega) = \inf\{\epsilon : \mu_\Omega(\epsilon) \geq 2^{-1}(1 - \gamma_m)|\Omega|\}. \quad (3)$$

We denote the number of Courant-sharp eigenvalues of  $\Omega$  by  $\mathfrak{C}(\Omega)$ .

**Theorem 1.** *Let  $\Omega$  be an open set in  $\mathbb{R}^m$  with finite Lebesgue measure. We have the following.*

(i) *If  $\lambda_n(\Omega)$  is Courant-sharp then*

$$\lambda_n(\Omega) \leq \left( \frac{2\pi m^2}{(1 - \gamma_m)\epsilon(\Omega)} \right)^2. \quad (4)$$

(ii)

$$\mathfrak{C}(\Omega) \leq \frac{\omega_m}{(1-\gamma_m)^m} (m^3(m+2))^{m/2} \frac{|\Omega|}{\epsilon(\Omega)^m}. \quad (5)$$

(iii) If  $n \in \mathbb{N}$ ,  $n > \frac{\omega_m}{(1-\gamma_m)^m} (m^3(m+2))^{m/2} \frac{|\Omega|}{\epsilon(\Omega)^m}$ , then  $\lambda_n(\Omega)$  is not Courant-sharp.

In Section 2 below we prove Theorem 1. In Section 3 we analyse some examples including the von Koch snowflake.

## 2 Proof of Theorem 1

Suppose  $\lambda_n(\Omega)$  is Courant-sharp with eigenfunction  $\varphi_{n,\Omega}$ . Let  $U_1, \dots, U_n$  be the nodal domains of  $\varphi_{n,\Omega}$  so that  $\lambda_n(\Omega) = \lambda_1(U_1) = \dots = \lambda_1(U_n)$ . Without loss of generality we may assume that  $|U_1| \leq |U_2| \leq \dots \leq |U_n|$ . Hence  $|U_1| \leq |\Omega|/n$ . By Faber-Krahn we have that

$$\lambda_n(\Omega) = \lambda_1(U_1) \geq \lambda_1(\mathcal{B}_m) \left( \frac{n\omega_m}{|\Omega|} \right)^{2/m}.$$

It follows that, since  $\lambda_{n-1}(\Omega) < \lambda_n(\Omega)$ ,

$$\begin{aligned} (\lambda_n(\Omega))^{m/2} &\geq (\lambda_1(\mathcal{B}_m))^{m/2} \frac{n\omega_m}{|\Omega|} \\ &\geq (\lambda_1(\mathcal{B}_m))^{m/2} \frac{\omega_m}{|\Omega|} (n-1) \\ &= (\lambda_1(\mathcal{B}_m))^{m/2} \frac{\omega_m}{|\Omega|} N_\Omega(\lambda_n(\Omega)). \end{aligned}$$

This gives that

$$\frac{\omega_m}{(2\pi)^m} (1-\gamma_m) |\Omega| (\lambda_n(\Omega))^{m/2} \leq R_\Omega(\lambda_n(\Omega)), \quad (6)$$

where  $R_\Omega : \mathbb{R}^+ \mapsto \mathbb{R}$  is defined by

$$R_\Omega(\lambda) = \frac{\omega_m}{(2\pi)^m} |\Omega| \lambda^{m/2} - N_\Omega(\lambda). \quad (7)$$

See (15) and (16) in [1]. Below we obtain an upper bound for  $R_\Omega(\lambda)$ . Let  $\epsilon > 0$  be arbitrary. Consider the collection  $\mathfrak{M}_\epsilon$  of open cubes of measure  $\epsilon^m$  with vertices in the set of  $m$ -tuples  $\{\mathbb{Z}\epsilon, \dots, \mathbb{Z}\epsilon\}$ . Let  $M_\Omega(\epsilon)$  be the number of open cubes of side-length  $\epsilon$  in  $\mathfrak{M}_\epsilon$  which are contained in  $\Omega$ ,

$$M_\Omega(\epsilon) = \#\{N \in \mathfrak{M}_\epsilon : N \subset \Omega\}.$$

We have that

$$|\Omega| - M_\Omega(\epsilon)\epsilon^m \geq 0. \quad (8)$$

In order to obtain an upper bound for the left hand-side of (8) we let  $x \in \Omega$ . If  $d(x, \partial\Omega) > m^{1/2}\epsilon$ , then  $x$  belongs to an open  $\epsilon$ -cube in  $\mathfrak{M}_\epsilon$  contained in  $\Omega$ .

Hence the measure of the set which is not covered by the  $\epsilon$ -cubes in  $\mathfrak{M}_\epsilon$  that are entirely contained in  $\Omega$  is bounded from above by  $\mu_\Omega(m^{1/2}\epsilon)$ . So

$$|\Omega| - M_\Omega(\epsilon)\epsilon^m \leq \mu_\Omega(m^{1/2}\epsilon). \quad (9)$$

By Dirichlet bracketing (see [15]) we have that

$$N_\Omega(\lambda) \geq M_\Omega(\epsilon)N_{C_\epsilon}(\lambda), \quad (10)$$

where  $C_\epsilon$  is an open cube in  $\mathbb{R}^m$  with side-length  $\epsilon$ . The following standard estimate is attributed to Gauss:

$$\begin{aligned} N_{C_\epsilon}(\lambda) &= \#\left\{(k_1, \dots, k_m) \in \mathbb{N}^m : \sum_{i=1}^m k_i^2 < \pi^{-2}\epsilon^2\lambda\right\} \\ &\geq \frac{\omega_m}{2^m} \left(\pi^{-1}\epsilon\lambda^{1/2} - m^{1/2}\right)_+^m \\ &\geq \frac{\omega_m}{(2\pi)^m} \epsilon^m \lambda^{m/2} \left(1 - \frac{\pi m^{3/2}}{\epsilon\lambda^{1/2}}\right), \end{aligned} \quad (11)$$

where  $+$  denotes the positive part. By (10) and (11),

$$\begin{aligned} N_\Omega(\lambda) &\geq M_\Omega(\epsilon)N_{C_\epsilon}(\lambda) \\ &\geq M_\Omega(\epsilon) \frac{\omega_m}{(2\pi)^m} \epsilon^m \lambda^{m/2} - M_\Omega(\epsilon) \frac{\omega_m}{(2\pi)^m} \pi m^{3/2} \epsilon^{m-1} \lambda^{(m-1)/2} \\ &= \frac{\omega_m}{(2\pi)^m} |\Omega| \lambda^{m/2} - (|\Omega| - M_\Omega(\epsilon)\epsilon^m) \frac{\omega_m}{(2\pi)^m} \lambda^{m/2} \\ &\quad - M_\Omega(\epsilon) \frac{\omega_m}{(2\pi)^m} \pi m^{3/2} \epsilon^{m-1} \lambda^{(m-1)/2}. \end{aligned} \quad (12)$$

We bound the second and third terms in the right hand-side of (12) using (9) and (8) respectively. This then gives, by (7), that

$$R_\Omega(\lambda) \leq \frac{\omega_m}{(2\pi)^m} \mu_\Omega(m^{1/2}\epsilon) \lambda^{m/2} + \frac{\pi m^{3/2} \omega_m}{(2\pi)^m} \frac{|\Omega| \lambda^{(m-1)/2}}{\epsilon}. \quad (13)$$

By (6) and (13) we have that if  $\lambda_n(\Omega)$  is Courant-sharp then

$$\begin{aligned} \frac{\omega_m}{(2\pi)^m} (1 - \gamma_m) |\Omega| (\lambda_n(\Omega))^{m/2} &\leq \frac{\omega_m}{(2\pi)^m} \mu_\Omega(m^{1/2}\epsilon) (\lambda_n(\Omega))^{m/2} \\ &\quad + \frac{\pi m^{3/2} \omega_m}{(2\pi)^m} \frac{|\Omega| (\lambda_n(\Omega))^{(m-1)/2}}{\epsilon}. \end{aligned} \quad (14)$$

We now choose  $\epsilon$  such that the second term in the right hand-side of (14) equals half of the left hand-side of (14). That is

$$\epsilon = 2\pi m^{3/2} (1 - \gamma_m)^{-1} (\lambda_n(\Omega))^{-1/2}. \quad (15)$$

By (14) and the choice of  $\epsilon$  in (15) we have that if  $\lambda_n(\Omega)$  is Courant-sharp then

$$2^{-1} (1 - \gamma_m) |\Omega| \leq \mu_\Omega(2\pi m^2 (1 - \gamma_m)^{-1} (\lambda_n(\Omega))^{-1/2}). \quad (16)$$

Since  $\epsilon \mapsto \mu_\Omega(\epsilon)$  is continuous and onto  $[0, |\Omega|]$  the infimum in (3) is over a non-empty set which is bounded from below, and therefore exists. So if  $\lambda_n(\Omega)$

is Courant-sharp then, by (3) and (16),  $\frac{2\pi m^2}{(1-\gamma_m)(\lambda_n(\Omega))^{1/2}} \geq \epsilon(\Omega)$ . This proves Theorem 1(i).

By [11] we also have that

$$\lambda_n(\Omega) \geq \frac{m}{m+2} \frac{(2\pi)^2}{\omega_m^{2/m}} \left( \frac{n}{|\Omega|} \right)^{2/m}. \quad (17)$$

This, together with (4), implies (5) and proves Theorem 1(ii).

To prove Theorem 1(iii) we just note that by (17),

$$\max \left\{ n \in \mathbb{N} : \lambda_n(\Omega) \leq \left( \frac{2\pi m^2}{(1-\gamma_m)\epsilon(\Omega)} \right)^2 \right\} \leq \frac{\omega_m}{(1-\gamma_m)^m} (m^3(m+2))^{m/2} \frac{|\Omega|}{\epsilon(\Omega)^m}.$$

□

We note that if we were to use the lower bounds for the counting function from Section 2 in [5] then we would have to assume a weak integrability condition on  $\mu_\Omega$  of the form  $\int \epsilon^{-1} d\mu_\Omega(\epsilon) < \infty$ . Such an integrability condition may fail if the interior Minkowski dimension of  $\partial\Omega$  is equal to  $m$ . The procedure above avoids this integrability condition.

### 3 Examples

In this section we analyse three examples where explicit computations seem out of reach.

**Example 1.** Let  $\Omega$  be an open, bounded, convex set in  $\mathbb{R}^m$ . Let  $\mathcal{H}^{m-1}(\partial\Omega)$  denote the  $(m-1)$ -dimensional Hausdorff measure of  $\partial\Omega$ . Then

$$\mathfrak{C}(\Omega) \leq \frac{\omega_m}{(1-\gamma_m)^{2m}} (4m^3(m+2))^{m/2} \frac{(\mathcal{H}^{m-1}(\partial\Omega))^m}{|\Omega|^{m-1}}. \quad (18)$$

*Proof.* By convexity of  $\Omega$  we have that

$$\mu_\Omega(\epsilon) \leq \mathcal{H}^{m-1}(\partial\Omega)\epsilon.$$

By (3),

$$\epsilon(\Omega) \geq 2^{-1}(1-\gamma_m) \frac{|\Omega|}{\mathcal{H}^{m-1}(\partial\Omega)}, \quad (19)$$

and (18) follows from Theorem 1 and (19). □

It was shown in [10] that only the first, second and fourth Dirichlet eigenvalues for  $\mathcal{B}_2$  are Courant-sharp. Hence  $\mathfrak{C}(\mathcal{B}_2) = 3$ , and the largest Courant-sharp eigenvalue for  $\mathcal{B}_2$  is equal to  $j_{0,2}^2$ . Here  $j_{0,2} \asymp 5.520..$  is the second positive zero of the Bessel function  $J_0$ . A straightforward computation using (4) and (19) shows that the largest Courant-sharp eigenvalue of  $\mathcal{B}_2$  is strictly less than  $1.2 \cdot 10^6$ . This compares well with the bound  $7.1 \cdot 10^6$  obtained in [1]. For the unit square  $\mathcal{C}_2$  it is known ([13], [2]) that only the first, second and fourth Dirichlet eigenvalues are Courant-sharp. Hence  $\mathfrak{C}(\mathcal{C}_2) = 3$ , and the largest Courant-sharp eigenvalue for  $\mathcal{C}_2$  is equal to  $8\pi^2$ . Using (4) and (19) we have that the largest Courant-sharp eigenvalue of the unit square is strictly less than  $4.5 \cdot 10^6$ , whereas

[1] gives a bound  $5.9 \cdot 10^6$ . These examples illustrate that the bounds obtained in Theorem 1 are very crude.

The second example is a von Koch snowflake  $K$  with similarity ratio  $\frac{1}{3}$ . We recall its construction. Let the basic square (generation 0) in  $K$  have side-length 1. The first generation consists of 4 squares with side-length  $\frac{1}{3}$  each attached symmetrically to the basic square. Proceeding inductively we have that the  $j$ 'th generation in  $K$ ,  $j \in \mathbb{N}$  consists of  $4 \cdot 5^{j-1}$  squares with side-length  $3^{-j}$ . We let  $K$  be the interior of its closure. Then  $K$  is connected, has Lebesgue measure  $|K| = 2$ , and both the Hausdorff dimension of  $\partial K$  and the interior Minkowski dimension of  $\partial K$  are equal to  $\log 5 / \log 3$ . See Figure 1, and [4] for further details.

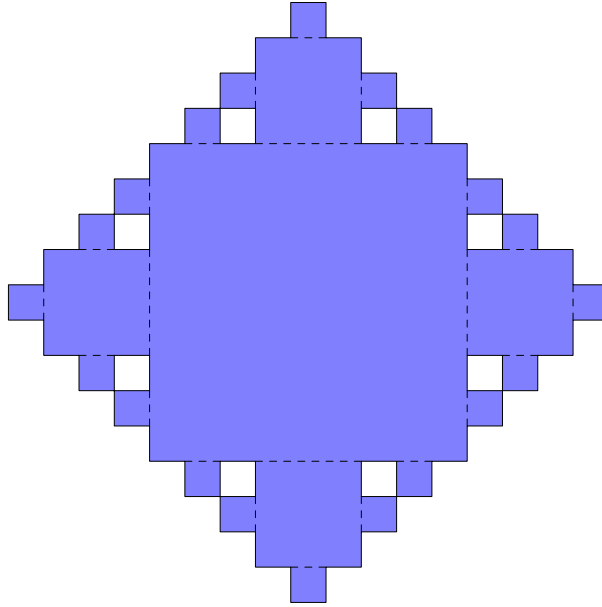


Figure 1: The first two generations of  $K$

**Example 2.** Let  $K$  be the von Koch snowflake generated by the unit square and similarity ratio  $\frac{1}{3}$ . Then

$$\mathfrak{C}(K) \leq 15 \cdot 10^7. \quad (20)$$

*Proof.* By Theorem 1, (2), and  $|K| = 2$  we find that

$$\mathfrak{C}(K) \leq \frac{64\pi j_0^4}{(j_0^2 - 4)^2} \epsilon(K)^{-2}, \quad (21)$$

where we have used that

$$\lambda_1(\mathcal{B}_2) = j_0^2,$$

where  $j_0 = 2.405\dots$  is the first positive zero of the Bessel function  $J_0$ . It remains to find a lower bound for  $\epsilon(K)$ . We obtain an upper bound for  $\mu_\Omega(\epsilon)$  by adding all edges between squares of different generations. This gives a disjoint union of 1 unit square and  $4 \cdot 5^{j-1}$  squares with side-lengths  $3^{-j}$ ,  $j \in \mathbb{N}$ . Let  $\epsilon < \frac{1}{18}$ ,

and let  $J \in \mathbb{N}$  be such that

$$J < \frac{\log\left(\frac{1}{2\epsilon}\right)}{\log 3} \leq J + 1.$$

Then  $J \geq 2$ . The contribution to the upper bound for  $\mu_\Omega(\epsilon)$  from the squares in generations  $1, \dots, J-1$  is bounded from above by

$$\left(4 + 16 \sum_{j=1}^{J-1} 5^{j-1} 3^{-j}\right) \epsilon \leq \frac{24\epsilon}{5} \left(\frac{5}{3}\right)^J \leq \frac{48}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon^{2-\frac{\log 5}{\log 3}}. \quad (22)$$

The first term in the left-hand side above is the contribution from the unit square. The contribution to the upper bound for  $\mu_\Omega(\epsilon)$  from the squares in generations  $J, J+1, \dots$  is bounded from above by

$$\sum_{j \geq J} 4 \cdot 5^{j-1} 9^{-j} = \left(\frac{5}{9}\right)^{J-1} \leq \frac{36}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon^{2-\frac{\log 5}{\log 3}}. \quad (23)$$

We recognise the interior Minkowski dimension  $\frac{\log 5}{\log 3}$  of  $\partial K$ . By (22) and (23) we have that

$$\mu_\Omega(\epsilon) \leq \frac{84}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon^{2-\frac{\log 5}{\log 3}}, \quad 0 < \epsilon < \frac{1}{18}.$$

Solving the equation

$$\frac{84}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon^{2-\frac{\log 5}{\log 3}} = 1 - \frac{4}{j_0^2}$$

gives that

$$\epsilon(K) \geq 0.00379. \quad (24)$$

The bound of (20) follows by (21) and (24).  $\square$

Below we construct an open set  $D_s \subset \mathbb{R}^3$ . Let  $Q_0 \subset \mathbb{R}^3$  be an open cube of side-length 1. Let  $0 < s \leq \sqrt{2} - 1$ . Attach a regular open cube  $Q_{1,i}$  of side-length  $s$  to the centre  $c_{1,i}, i = 1, \dots, 6$ , of each face of  $\partial Q_0$ , and such that all the faces are pairwise-parallel. Now proceed by induction. For  $j = 2, 3, \dots$ , attach  $N(j) = 6 \cdot 5^{j-1}$  open cubes  $Q_{j,1}, \dots, Q_{j,N(j)}$ , of side-length  $s^j$  to the centres of the boundary faces of the cubes  $Q_{j-1,1}, \dots, Q_{j-1,N(j-1)}$ , again with pairwise-parallel faces. We define the polyhedron  $D_s$  as

$$D_s = \text{interior} \left\{ \overline{Q_0 \cup \left[ \bigcup_{j \geq 1} \bigcup_{1 \leq i \leq N(j)} Q_{j,i} \right]} \right\}.$$

See Figure 2. We note that for  $0 < s \leq \sqrt{2} - 1$  no cubes in the construction of  $D_s$  overlap.

The asymptotic behaviour of the heat content of  $D_s$  in  $\mathbb{R}^3$  for small time was analysed in [6]. Here we have the following.

**Example 3.** Let  $s \in (0, \sqrt{2} - 1]$ , and let  $D_s$  be the polyhedron in  $\mathbb{R}^3$  defined above. Then

$$\mathfrak{C}(D_s) \leq 25 \cdot 10^{10}. \quad (25)$$

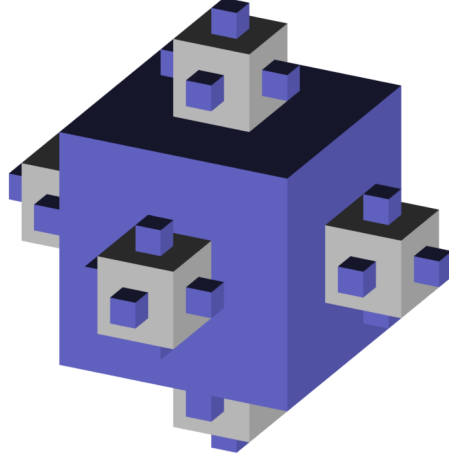


Figure 2: The first two generations of  $D_s$  with  $s = \frac{1}{3}$ .

*Proof.* We have that

$$|D_s| = \frac{1 + s^3}{1 - 5s^3},$$

and that the two-dimensional Hausdorff measure of the boundary is given by

$$\mathcal{H}^2(\partial D_s) = 6 \left( \frac{1 - s^2}{1 - 5s^2} \right).$$

By Theorem 1 we have that

$$\mathfrak{C}(D_s) \leq 36(15)^{3/2}\pi \left(1 - \frac{9}{2\pi^2}\right)^{-3} \frac{|D_s|}{\epsilon(D_s)^3}, \quad (26)$$

where we have used that

$$\lambda_1(\mathcal{B}_3) = j_{1/2}^2 = \pi^2,$$

where  $j_{1/2} = \pi$  is the first positive zero of the Bessel function  $J_{1/2}$ . We obtain an upper bound for  $\mu_\Omega(\epsilon)$  by adding all faces between cubes of different generations. This gives a disjoint union of 1 unit cube and  $6 \cdot 5^{j-1}$  cubes of side-length  $s^j$ ,  $j \in \mathbb{N}$ . Hence

$$\mu_\Omega(\epsilon) \leq \left(6 + 36 \sum_{j=1}^{\infty} 5^{j-1} s^{2j}\right) \epsilon = \frac{6(1 + s^2)}{1 - 5s^2} \epsilon. \quad (27)$$

By (3) and (27) we have that

$$\epsilon(D_s) \geq \frac{1}{12} \left(1 - \frac{9}{2\pi^2}\right) \frac{1 - 5s^2}{1 + s^2} |D_s|. \quad (28)$$

Finally by (26), (28), the fact that  $0 < s \leq \sqrt{2} - 1$ , and  $|D_s| \geq 1$  we obtain that

$$\mathfrak{C}(D_s) \leq 6(12)^4 (15)^{3/2} (140 + 99\sqrt{2}) \pi \left(1 - \frac{9}{2\pi^2}\right)^{-6}.$$

This implies (25). □



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